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LETTER TO THE EDITOR

A (p, q)-oscillator realization of two-parameter quantum algebras

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Abstract. It is noted that the study of a quantum algebra $su_{p,q}(2)$, with two independent deformation parameters (p, q), leads to a '(p, q)-oscillator' realization for it. The analysis is extended to the (p, q)-analogues of su(1, 1), osp(2|1) and the centreless Virasoro algebra. The standard single-parameter (q) deformations are obtained in the limit $p = q$.

Quantum algebras [1, 2] arise as the underlying mathematical structure in several contexts like quantum inverse scattering theory, solutions of the Yang–Baxter equation, and rational conformal field theory [3, 4]. These algebras may be viewed as deformations of classical Lie algebras, depending, in general, on one or more parameters. The representation theory of quantum algebras with a single deformation (or quantization) parameter q, has led to the development of q-deformed oscillator algebras [5–10]. Similar deformed oscillator algebras have been studied earlier [11–13] with a view to exploring new quantization procedures. These q-oscillators may lead to a new kind of field theory where a small violation of the Pauli exclusion principle and deviations from the Bose statistics may be discussed [14–16]. In addition, the q-analogues of the parabose and parafermi oscillators [17] and the supersymmetric quantum mechanical algebras [18, 19] have been considered. The implications of q-deformed algebraic structures in concrete physical models such as the Jaynes–Cummings model in quantum optics have been investigated [20].

From the point of view of applicability in concrete physical models, quantum algebras with multiparameter deformations [21–23] are of interest. But it has been argued [24, 25] that any quantum algebra with one or more deformation parameters may be mapped onto the standard single-parameter case. Recently, while studying a two-parameter (p, q) deformation of GL(2) it has been noted [26] that the corresponding quantum algebra $gl_{p,q}(2)$ may be mapped onto the standard deformation of gl(2) with a single parameter equal to $\sqrt{qp}$. However, significantly, it has been emphasized in [26] that p and q are two genuinely independent quantization parameters as exhibited by the comultiplication rules and the structure of the endomorphisms of the quantum group acting on the underlying non-commutative space as embodied in the R-matrix. Now, knowing the convenience of the language of q-oscillators in describing the representations of the quantum algebras with a single deformation parameter q, it is natural to seek a '(p, q)-oscillator' realization of the (p, q)-deformed algebras. Here, we first notice that there is a (p, q)-oscillator realization for a (p, q)-deformed su(2); then, the analysis is extended to the (p, q)-analogues of su(1, 1), osp(2|1) and the centreless Virasoro algebra.

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To define an $su_{p,q}(2)$ algebra, we start, as implied by [26], by considering the standard deformation of su(2) with a single parameter equal to $\sqrt{qp}$

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{(qp)^{J_0} - (qp)^{-J_0}}{(qp)^{1/2} - (qp)^{-1/2}} = [2J_0]_{(qp)^{1/2}}. \tag{1}$$

Using a map $J \to \tilde{J}$ defined by

$$\tilde{J}_0 = J_0 \qquad \tilde{J}_+ = (\tilde{J}_-)^\dagger = (qp^{-1})^{(1/2)(J_0 - 1/2)} J_+ \tag{2}$$

we obtain the commutation relations

$$[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm \qquad \tilde{J}_+\tilde{J}_- - q^{-1}p\tilde{J}_-\tilde{J}_+ = [2\tilde{J}_0]_{p,q} \tag{3}$$

where we define

$$[X]_{p,q} = \frac{q^X - p^{-X}}{q - p^{-1}}. \tag{4}$$

The commutation relations (3) constitute an $su_{p,q}(2)$ algebra with $(\tilde{J}_0, \tilde{J}_\pm)$ as the generators. Note that in the limit $p = q$, $[X]_{p,q} \to [X]_q$ and $su_{p,q}(2) \to su_q(2)$. The coproduct rules

$$\Delta(\tilde{J}_0) = \tilde{J}_0 \otimes 1 + 1 \otimes \tilde{J}_0 \qquad \Delta(\tilde{J}_\pm) = q^{\tilde{J}_0} \otimes \tilde{J}_\pm + \tilde{J}_\pm \otimes p^{-\tilde{J}_0} \tag{5}$$

refer to an algebra homomorphism for (3). The $q \leftrightarrow p^{-1}$ symmetry in (3), and consequently in (5), reduces to the $q \leftrightarrow q^{-1}$ symmetry in the case of $su_q(2)$.

The commutation relations (3) may be described in terms of an R-matrix [3, 26]. With

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & qp^{-1} & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \lambda = q - p^{-1} \tag{6}$$

a constant solution to the Yang–Baxter equation, the relations (3) translate to

$$R(L^{(\varepsilon_1)} \otimes 1)(1 \otimes L^{(\varepsilon_2)}) = (1 \otimes L^{(\varepsilon_2)})(L^{(\varepsilon_1)} \otimes 1)R \tag{7}$$

where $(\varepsilon_1, \varepsilon_2) = (+,+), (-,+), (-,-)$ and

$$L^{(+)} = \begin{pmatrix} q^{\tilde{J}_0} & \lambda q^{-1} p \tilde{J}_+ \\ 0 & p^{-\tilde{J}_0} \end{pmatrix} \qquad L^{(-)} = \begin{pmatrix} p^{-\tilde{J}_0} & 0 \\ -\lambda \tilde{J}_- & q^{\tilde{J}_0} \end{pmatrix}. \tag{8}$$

The coproduct is determined by

$$\Delta(L^{(\pm)}) = L^{(\pm)} \dot{\otimes} L^{(\pm)} \tag{9}$$

where $\dot{\otimes}$ denotes the tensor product combined with the usual matrix multiplication. As emphasized in [26], it is seen that the deformation parameters p and q may be varied independently. Hereafter, $[X]_{p,q}$ defined by (4) for any X, will be denoted, in general, simply as $[X]$; similarly, in other (p, q)-related expressions the 'p, q' index will not be, in general, exhibited explicitly.

Following [24, 25], the nonlinear maps relating the generators $(j_0, j_+, j_- = (j_+)^\dagger)$ of the classical su(2) to the corresponding elements of $su_{p,q}(2)$ may be defined as

$$\tilde{J}_0 = j_0 \qquad \tilde{J}_+ = (\tilde{J}_-)^\dagger = \left\{ \frac{(q^{-1}p)^{j-j_0}[j - j_0 + 1][j + j_0]}{(j - j_0 + 1)(j + j_0)} \right\}^{1/2} j_+ \tag{10}$$

for an integral or half-integral j. Now, the $(2j+1)$-dimensional representation follows immediately

$$\tilde{J}_0|j, m\rangle = m|j, m\rangle \qquad m = j, j-1, \ldots, -j$$
$$\tilde{J}_\pm|j, m\rangle = \{(q^{-1}p)^{j-m-(1\pm1)/2}[j\mp m][j\pm m+1]\}^{1/2}|j, m\pm1\rangle. \tag{11}$$

To check that this representation satisfies the algebra (3) we use the identity

$$(q^{-1}p)^{n_2}\{[n_1][n_2+1]-[n_2][n_1+1]\} = [n_1-n_2]. \tag{12}$$

In the limit $p = q$ this identity reduces to the familiar relation

$$[n_1]_q[n_2+1]_q - [n_2]_q[n_1+1]_q = [n_1-n_2]_q. \tag{13}$$

Except when p and q are such that $p^{-n} = q^n$ for some $n > 0$, $su_{p,q}(2)$, $su_{\sqrt{qp}}(2)$ and su(2) provide nonlinear realizations of each other and their representation theories have close kinship. We take, throughout, p and q to be real, and such, that $[n] > 0$ for any $n > 0$, unless otherwise specified.

Let us now consider the representation (11) in the limit $j \to \infty$. Defining

$$N = \lim_{j\to\infty}(j-\tilde{J}_0) \qquad A = (A^\dagger)^\dagger = \lim_{j\to\infty}(qp^{-1})^{(j-\tilde{J}_0)/2}[j+\tilde{J}_0]^{-1/2}\tilde{J}_+ \tag{14}$$

it is found that the spectrum of N becomes $(0, 1, 2, \ldots)$ and

$$[N, A] = -A \qquad [N, A^\dagger] = A^\dagger \tag{15a}$$

$$AA^\dagger - qA^\dagger A = p^{-N} \tag{15b}$$

$$AA^\dagger - p^{-1}A^\dagger A = q^N. \tag{15c}$$

To see this, one has to note that in the representation (11)

$$AA^\dagger = [N+1] \qquad A^\dagger A = [N] \tag{16}$$

and the definition (4) implies

$$[n+1] = q[n] + p^{-n} = p^{-1}[n] + q^n \qquad \text{for } n = 0, 1, 2, \ldots. \tag{17}$$

It is natural to identify A, $A^\dagger$ and N, obeying the commutation relations (15), respectively as the annihilation, creation and the excitation number operators of a '(p, q)-oscillator' since in the limit $p = q$ the relations characterize the q-oscillator. Thus, we notice that the $su_{p,q}(2)$ algebra (3) leads to a (p, q)-oscillator in the same way as the $su_q(2)$ algebra leads to the q-oscillator under contraction [10]. It may be noted that the relations (15b) and (15c) imply each other and this $q \leftrightarrow p^{-1}$ symmetry generalizes the $q \leftrightarrow q^{-1}$ symmetry of the q-oscillator.

The Fock space representation of a single-mode (p, q)-oscillator may be specified as follows. Let a, $a^\dagger$ and N denote, generally, the annihilation, creation and the number operators. With $\{|n\rangle \,|\, n = 0, 1, 2, \ldots\}$ as the complete orthonormal set of eigenstates of N, one has, in general,

$$a|0\rangle = 0 \qquad N|n\rangle = n|n\rangle \qquad a|n\rangle = \alpha(n)|n-1\rangle \qquad a^\dagger|n-1\rangle = \bar{\alpha}(n)|n\rangle$$
$$aa^\dagger = |\alpha(N+1)|^2 \qquad a^\dagger a = |\alpha(N)|^2 \tag{18}$$
$$|n\rangle = \{\bar{\alpha}(n)!\}^{-1}(a^\dagger)^n|0\rangle \qquad \bar{\alpha}(n)! = \prod_{k=1}^{n}\bar{\alpha}(k)$$

where the sequence $\{|\alpha(n)|^2|n=0,1,2,\ldots;\alpha(0)=0\}$ characterizes the particular system. For the (p,q)-oscillator we have $\{|\alpha(n)|^2=[n]\}$. It is interesting to note that $[n]$ is the unique solution to the generalized Fibonacci recursion relation [27]

$$[n+1]=(q+p^{-1})[n]-qp^{-1}[n-1] \qquad n\geqslant 1 \qquad [1]=1 \qquad [0]=0. \tag{19}$$

Extending the relations (15*b*, *c*) we can write, for $n\geqslant 1$,

$$A(A^\dagger)^n=q^n(A^\dagger)^nA+[n](A^\dagger)^{n-1}p^{-N}=p^{-n}(A^\dagger)^nA+[n](A^\dagger)^{n-1}q^N. \tag{20}$$

Analogous to the familiar boson realization of the q-oscillator

$$A=\left(\frac{[N+1]}{N+1}\right)^{1/2}b \qquad A^\dagger=b^\dagger\left(\frac{[N+1]}{N+1}\right)^{1/2} \qquad N=b^\dagger b \tag{21}$$

for the (p,q)-oscillator, with $(b,b^\dagger)$ as the usual boson operators.

As already noted, p and q may be varied independently. For the boson $(|\alpha(n)|^2=n)$ $p=q=1$. For the fermion with $(q=-1, p=1)$ the sequence $\{|\alpha(n)|^2=[n]_{1,-1}\}$ has the first zero, after [0], at $n=2$ in accordance with the Pauli principle $A^\dagger|1\rangle=0$. It may be noted that, due to the $q\leftrightarrow p^{-1}$ symmetry, the fermion can also be described as a $(q=1, p=-1)$-oscillator as is verified directly by using the standard matrix representation. When $q=p$ $(-p)$ the relations (15) reduce to the commutation properties of bosonic (fermionic) q-oscillators [5-10, 18, 19]. The choice $p=1$, with arbitrary q, corresponds to the deformed oscillators studied in [11-13]. The example $(q=0, p^{-1}\neq 0)$ (or $q\neq 0$, $p^{-1}=0$) gives a deformation of a single mode of the oscillators exhibiting 'infinite statistics' [14]. Hereafter, by (p,q)-oscillator we refer to the generic case (15) with arbitrary p and q unless otherwise specified.

A representation of $\mathrm{su}_{p,q}(2)$ constructed from two mutually commuting sets of (p,q)-boson operators $(A_1, A_1^\dagger, N_1)$ and $(A_2, A_2^\dagger, N_2)$, *à la* Jordan-Schwinger, may be given as

$$\tilde{J}_0=\tfrac{1}{2}(N_1-N_2) \qquad \tilde{J}_+=(\tilde{J}_-)^\dagger=A_1^\dagger(q^{-1}p)^{N_2/2}A_2. \tag{22}$$

The weight vectors $\{|j,m\rangle|-j\leqslant m\leqslant j\}$ carrying the $(2j+1)$-dimensional representation (11) are now

$$|j,m\rangle=\{\bar{\alpha}(j+m)!\,\bar{\alpha}(j-m)!\}^{-1}(A_1^\dagger)^{j+m}(A_2^\dagger)^{j-m}|0,0\rangle \qquad |\alpha(n)|^2=[n]. \tag{23}$$

The Casimir invariant in both its antipodal forms is

$$\begin{aligned} C&=(q^{-1}p)^{\tilde{J}_0}([\tilde{J}_0][\tilde{J}_0+1]+q^{-1}p\tilde{J}_-\tilde{J}_+)\\ &=(q^{-1}p)^{\tilde{J}_0}(\tilde{J}_+\tilde{J}_-+qp^{-1}[\tilde{J}_0][\tilde{J}_0-1]). \end{aligned} \tag{24}$$

The eigenvalues of C are given by

$$C|j,m\rangle=(q^{-1}p)^j[j][j+1]|j,m\rangle. \tag{25}$$

To facilitate the extension of the traditional q-analysis [28] to a (p,q)-analysis we make the following preliminary observations:

$$[n]|_{p^{-1}=q}=nq^{n-1} \tag{26a}$$

$$[-1]=-q^{-1}p \qquad [-n]=-(q^{-1}p)^n[n]. \tag{26b}$$

A generating function for $[n]$ is

$$\sum_{n=0}^{\infty}[n]z^n=z\{(1-qz)(1-p^{-1}z)\}^{-1}. \tag{27}$$

Defining

$$\tilde{D}_z\psi(z)=\frac{\psi(qz)-\psi(p^{-1}z)}{qz-p^{-1}z} \tag{28}$$

we have

$$\tilde{D}_z z^n=[n]z^{n-1} \tag{29}$$

and

$$\frac{z^{n+1}}{[n+1]}=\begin{cases}(q-p^{-1})z\sum_{k=0}^{\infty}q^{-(k+1)}p^{-k}(q^{-(k+1)}p^{-k}z)^n & \text{for } |qp|>1\\ (p^{-1}-q)z\sum_{k=0}^{\infty}p^{(k+1)}q^k(p^{(k+1)}q^kz)^n & \text{for } |qp|<1.\end{cases} \tag{30}$$

The (p,q)-exponential is defined by

$$\exp_{p,q}(z)=\sum_{n=0}^{\infty}z^n/[n]! \qquad \tilde{D}_z\exp_{p,q}(\mu z)=\mu\exp_{p,q}(\mu z). \tag{31}$$

This leads to the identification of the coherent states of the (p,q)-oscillator $\{|z\rangle_{p,q}\,|A|z\rangle_{p,q}=z|z\rangle_{p,q}\}$ as

$$\begin{aligned}|z\rangle_{p,q}&=N(z)\{\exp_{p,q}(zA^\dagger)\}|0\rangle \qquad N(z)=\{\exp_{p,q}(|z|^2)\}^{-1/2}\\ \langle\zeta|z\rangle&=N(\zeta)N(z)\exp_{p,q}(\bar{\zeta}z).\end{aligned} \tag{32}$$

The (p,q)-deformation of the Bargmann–Fock representation closely follows the q-deformed case [29]. In the space of analytic functions of the complex variable z one has the correspondence

$$A\to\tilde{D}_z \qquad A^\dagger\to z \qquad N\to z\frac{\partial}{\partial z} \tag{33}$$

and the inner product which makes z and $\tilde{D}_z$ Hermitian conjugates is

$$(f,g)=f(\tilde{D}_z)g(z)|_{z=0}. \tag{34}$$

The set of functions $\{\langle n|z\rangle=z^n/([n]!)^{1/2}|n=0,1,2,\ldots\}$ forms an orthonormal basis with respect to this inner product.

It is well known [30] that in the undeformed case the parafermi cretation and annihilation operators form realizations of su(2). This characteristic has been exploited [17] to define the q-analogue of the parafermi oscillator. To specify a single-mode (p,q)-parafermion of order $g=2j$ ($j=1,\frac{3}{2},\ldots;\ j=\frac{1}{2}$ corresponds to the fermion) we identify the corresponding annihilation (F), creation ($F^\dagger$) and the number (N) operators respectively with the $(2j+1)$-dimensional representations of $\tilde{J}_+,\tilde{J}_-$ and $(j-\tilde{J}_0)$ of $\mathrm{su}_{p,q}(2)$. The Fock space representation is given by (16) with the identification $(a=F,\ a^\dagger=F^\dagger,\ (\alpha(n)=\{(q^{-1}p)^{n-1}[n][g+1-n]\}^{1/2}|n=0,1,2,\ldots,g))$. From the $\mathrm{su}_{p,q}(2)$ algebra relations (3) and the properties of $[n]$ like (17) it is straightforward to generate the (p,q)-extension of the characteristic triple commutation relations of the parafermi algebra. Note that when $g=1$ the (p,q)-parafermion is identical to the usual fermion independent of p and q. For $g>1$ the (p,q)-parafermion reduces to the usual parafermion in the limit $p=q=1$.

To obtain the $\mathrm{su}_{p,q}(1,1)$ algebra from $\mathrm{su}_{p,q}(2)$, we mimic the well known $\mathrm{su}(2)\leftrightarrow\mathrm{su}(1,1)$ relation. We define

$$\tilde{K}_0=\tilde{J}_0 \qquad \tilde{K}_+=\mathrm{i}(q^{-1}p)^{1/2}\tilde{J}_+ \qquad \tilde{K}_-=\mathrm{i}(q^{-1}p)^{1/2}\tilde{J}_- \qquad q,p>0. \tag{35}$$

Note that $\mathrm{i}(q^{-1}p)^{1/2}=[-1]^{1/2}$ when $q,p>0$. Now, $(\tilde{K}_0,\tilde{K}_\pm)$ generate the $\mathrm{su}_{p,q}(1,1)$ algebra

$$[\tilde{K}_0,\tilde{K}_\pm]=\pm\tilde{K}_\pm \qquad \tilde{K}_-\tilde{K}_+-qp^{-1}\tilde{K}_+\tilde{K}_-=[2\tilde{K}_0] \tag{36}$$

with the coproduct rules

$$\Delta(\tilde{K}_0)=\tilde{K}_0\otimes 1+1\otimes\tilde{K}_0 \qquad \Delta(\tilde{K}_\pm)=q^{\tilde{K}_0}\otimes\tilde{K}_\pm+\tilde{K}_\pm\otimes p^{-\tilde{K}_0} \tag{37}$$

following readily from (5) and (35). The Casimir operator is

$$C=(q^{-1}p)^{\tilde{K}_0}(\tilde{K}_-\tilde{K}_+-[\tilde{K}_0][\tilde{K}_0+1])=(q^{-1}p)^{\tilde{K}_0-1}(\tilde{K}_+\tilde{K}_--[\tilde{K}_0][\tilde{K}_0-1]). \tag{38}$$

Substituting the $(2j+1)$-dimensional representations (11) for $\mathrm{su}_{p,q}(2)$ in (35) one obtains the finite-dimensional non-Hermitian representations of $\mathrm{su}_{p,q}(1,1)$. In the jth representation C takes the value $[-j][j+1]$.

Hermitian realization of $\mathrm{su}_q(1,1)$ (which is $\mathrm{su}_{q,q}(1,1)$) has been constructed [20] by q-deforming a special case of the Holstein–Primakoff representation. An extension of this procedure to obtain a Hermitian realization of $\mathrm{su}_{p,q}(1,1)$ is straightforward:

$$\tilde{K}_0=N+\tfrac{1}{2} \qquad \tilde{K}_+=(\tilde{K}_-)^\dagger=[N]^{1/2}A^\dagger. \tag{39}$$

To verify that this realization satisfies the desired algebra relations (36) we use the identity

$$[n+1]^2-qp^{-1}[n]^2=[2n+1]. \tag{40}$$

In the representation (39) C has the value $[1/2]^2$.

To construct a single mode (p,q)-parabose oscillator of order $g=2,3,\ldots$ we generalize the well known connection [30] between the undeformed parabose operators and $\mathrm{su}(1,1)$. Choosing in (16) $\{|\alpha(2\nu)|^2=[2\nu],\ |\alpha(2\nu+1)|^2=[2\nu+g]\,|\,\nu=0,1,2,\ldots\}$ the corresponding (p,q)-parabose operators, B (annihilation) and $B^\dagger$ (creation), satisfy the relations

$$[N,B]=-B \qquad [N,B^\dagger]=B^\dagger \qquad [1/2]\{BB^\dagger+(qp^{-1})^{1/2}B^\dagger B\}=[N+g/2]. \tag{41}$$

As in the parafermion case, (p,q)-generalized triple commutation relations for $(B,B^\dagger,N)$ can be generated using the properties of $[n]$. When $g=1$, it is evident that the (p,q)-paraboson becomes the (p,q)-boson. In the limit $p=q$, q-parabosons [17] are obtained.

The identification

$$\tilde{K}_0=\tfrac{1}{2}(N+g/2) \qquad \tilde{K}_+=(\tilde{K}_-)^\dagger=[2]^{-1}(B^\dagger)^2 \tag{42}$$

leads to a realization of $\mathrm{su}_{p^2,q^2}(1,1)$ in view of the identity

$$\begin{aligned}
&[n+g][n+2]-(qp^{-1})^2[n+g-2][n]\\
&\qquad=[n+g+1][n+1]-(qp^{-1})^2[n+g-1][n-1]\\
&\qquad=[2]^2[n+g/2]_{p^2,q^2}.
\end{aligned} \tag{43}$$

In this realization C takes the value $([g/4]_{p^2,q^2}[1-g/4]_{p^2,q^2})$.

The relations (41), (42) are seen to generate a (p, q)-extension of the quantum superalgebra $\mathrm{osp}_q(2|1)$ [31]. With $V_- = B$, $V_+ = B^\dagger$, we have a (p, q)-analogue of the graded su(1, 1)

$$\begin{aligned} &\{V_- V_+ + (qp^{-1})^{1/2} V_+ V_-\} = [1/2]^{-1}[2\tilde{K}_0] \qquad [\tilde{K}_0, V_\pm] = \pm\tfrac{1}{2} V_\pm \\ &\tilde{K}_\mp V_\pm - (qp^{-1})^{\pm 1} V_\pm \tilde{K}_\mp = ([\pm 1]/[2])(q^{2\tilde{K}_0 \pm 1/2} + p^{-2\tilde{K}_0 \mp 1/2}) V_\mp \qquad (44) \\ &\tilde{K}_- \tilde{K}_+ - (qp^{-1})^2 \tilde{K}_+ \tilde{K}_- = [2\tilde{K}_0]_{p^2, q^2}. \end{aligned}$$

Using the $\mathrm{su}(2) \leftrightarrow \mathrm{su}(1, 1)$ connection (35) one can get a (p, q)-analogue of the graded su(2). In the limit $p = q$, $\mathrm{osp}_{p,q}(2|1) \to \mathrm{osp}_q(2|1)$. When $p = q = 1$ the above results lead to the paraboson realizations of osp(2|1) [32, 33].

Finally we mention that the (p, q)-oscillator algebra may be used to study a centreless (p, q)-Virasoro algebra. The analogous problem for the q-Virasoro algebra has been considered earlier [24, 34–36]. For the (p, q)-Virasoro algebra

$$L_n = (A^\dagger)^{n+1} A \qquad L_n L_m - q^{m-n} L_m L_n = [m - n] p^{-N+m} L_{m+n}. \qquad (45)$$

A more symmetric deformation is obtained by introducing the generators $\hat{L}_n = p^N L_n$:

$$p^{n-m} \hat{L}_n \hat{L}_m - q^{m-n} \hat{L}_m \hat{L}_n = [m - n] \hat{L}_{m+n} \qquad (46a)$$

$$[\hat{L}_n, \hat{L}_m] = [m - n] p^{N-n} q^{N-m} \hat{L}_{m+n}. \qquad (46b)$$

To obtain (46) we use the identity (20). The appearance of the ordinary commutator in (46b) immediately leads to the Jacobi identity

$$[\hat{L}_k, [\hat{L}_l, \hat{L}_m]] + \text{cyclic permutations} = 0. \qquad (47)$$

We notice, however, that in contradistinction to the corresponding single-parameter case [36] we do not find the generators $\hat{L}_n$ satisfying a deformed Jacobi identity. Using (20) and (46) we obtain a deformed su(1, 1) subalgebra of the centreless (p, q)-Virasoro algebra

$$\begin{aligned} &p^{-1} \hat{L}_0 \hat{L}_1 - q \hat{L}_1 \hat{L}_0 = \hat{L}_1 \qquad p^{-1} \hat{L}_{-1} \hat{L}_0 - q \hat{L}_0 \hat{L}_{-1} = \hat{L}_{-1} \\ &[\hat{L}_{-1}, \hat{L}_1] = (q^{-1} p)[2](\hat{L}_0 + (q - p^{-1}) \hat{L}_0^2). \end{aligned} \qquad (48)$$

In summary, we have derived a (p, q)-analogue of the q-boson oscillator from the study of a (p, q)-deformed su(2) algebra and used it to construct the realizations of $\mathrm{su}_{p,q}(2)$, $\mathrm{su}_{p,q}(1, 1)$, $\mathrm{osp}_{p,q}(2|1)$ and a centreless (p, q)-Virasoro algebra. The (p, q)-analogues of the fermionic, parafermionic and the parabosonic oscillators have also been identified.

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